Lecture 25

We would like to find antiderivatives

We need develop a way of evaluating integrals like

$$\int 2x\sqrt{1+x^2}\,dx$$

To evaluate this integral our strategy is to simplify the integrals by changing from the variable x to a new variable u.

$u = 1 + x^2 \Rightarrow$ the quantity under the square root

Then $\dfrac{du}{dx} = 2x \Rightarrow du = 2x\,dx$

$$\int \sqrt{1+x^2}\cdot 2x\,dx = \int \sqrt{u}\cdot du = \int u^{1/2}\,du = \frac{u^{3/2}}{3/2} + C$$

$$= \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+x^2)^{3/2} + C$$

In general, this works whenever we have an integral that we can write in the form $\int f(g(x)) \cdot g'(x)\, dx$.

Observe that if $F' = f$, then

$$\int F'(g(x)) \cdot g'(x)\, dx = ?$$

$$\frac{d}{dx}\left(F(g(x))\right) = F'(g(x)) \cdot g'(x)$$

So, $\int F'(g(x)) \cdot g'(x)\, dx = F(g(x)) + C$

If we let $u = g(x)$, $\frac{du}{dx} = g'(x) \Rightarrow du = g'(x) du$

We have the following rule:

If $u = g(x)$ is a diff. function (whose range is an interval I) and f is continuous on I, then

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du$$

Ex

$$\int x^3 \cos(x^4+4)\,dx$$

Let $u = x^4 + 4$; $\dfrac{du}{dx} = 4x^3 \Rightarrow du = 4x^3\,dx \Rightarrow \dfrac{du}{4} = x^3\,dx$

$$= \int \cos(x^4+4)\,x^3\,dx = \int \cos(u)\cdot\frac{1}{4}\,du = \frac{1}{4}\int \cos(u)\,du$$

$$= \frac{1}{4}\sin u + C = \frac{1}{4}\sin(x^4+4) + C.$$

Ex

$$\int \sqrt{2x+1}\,dx$$

let $u = 2x+1$

$\dfrac{du}{dx} = 2 \Rightarrow dx = \dfrac{1}{2}\,du$

$$\int \sqrt{u}\cdot\frac{1}{2}\,du = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\cdot\frac{2}{3}\,u^{3/2} + C$$

$$= \frac{1}{3}(2x+1)^{3/2} + C$$

Ex $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\,dx$

Let $u = 1-4x^2$

$$\frac{du}{dx} = -8x$$

$$\Rightarrow \frac{-du}{\quad} = x\,dx$$

$$\int \frac{-du/8}{\sqrt{u}} = -\frac{1}{8}\int u^{-1/2}\,du = -\frac{1}{8}\,\frac{u^{1/2}}{1/2} + C$$

$$= \frac{-u^{1/2}}{4} + C = -\frac{1}{4}\sqrt{1-4x^2} + C$$

Ex $\displaystyle\int \cos\theta \cdot \sin^6 d\theta = \int \cos\theta\,(\sin\theta)^6\,d\theta$

let $u = \sin\theta$

$$\frac{du}{d\theta} = \cos\theta \quad \Rightarrow du = \cos\theta\,d\theta$$

$$\int u^6\,du = \frac{u^7}{7} + C = \frac{\sin^7\theta}{7} + C$$

Ex $\displaystyle\int x^2 (x^3+5)^9 \, dx$

Let $u = x^3+5$, $\dfrac{du}{dx} = 3x^2 \Rightarrow \dfrac{du}{3} = x^2 dx$

$$\int \frac{u^9}{3} \, du = \frac{u^{10}}{30} + C = \frac{(x^3+5)^{10}}{30} + C$$

Ex $\displaystyle\int \cos 5\varphi \, d\varphi$

Let $u = 5\varphi$ $\dfrac{du}{d\varphi} = 5 \Rightarrow d\varphi = \dfrac{du}{5}$

$$\int \cos u \cdot \frac{du}{5} = \frac{1}{5}\int \cos u \, du = \frac{1}{5}\sin u + C = \frac{1}{5}\sin(5\varphi) + C$$

Definite Integrals

Two methods are possible

Ex

$$\int_0^7 \sqrt{4+3x} \, dx$$

Method 1

Evaluate the indefinite integral first and then use Evaluation Theorem

$$\int_0^7 \sqrt{4+3x}\,dx = \int \sqrt{4+3x}\,dx \Big]_0^7$$

Let $u = 4 + 3x$

$$\frac{du}{dx} = 3 \Rightarrow \frac{1}{3}\,du = dx$$

$$\int \sqrt{u}\,\frac{du}{3} \Big]_{x=0}^{x=7} = \frac{2}{9}u^{3/2} \Big]_{x=0}^{x=7} = \frac{2}{9}(4+3x)^{3/2} \Big]_0^7$$

$$= \frac{2}{9}(25)^{3/2} - \frac{2(4)^{3/2}}{9} = \frac{250-16}{9} = \frac{234}{9} = 26$$

Method 2

Usually easier is to change the limits of integration when the variable is changed.

Let $u = 4 + 3u$ When $x = 0$, $u = 4$

$\frac{du}{3} = dx$ $x = 7$, $u = 25$

So when we rewrite evth in terms of du, we get

$$\int_4^{25} \sqrt{u}\ \frac{du}{3} = \frac{1}{3}\frac{u^{3/2}}{3/2}\Bigg]_4^{25} = \frac{1}{3}\cdot\frac{25^{3/2}}{3/2} - \frac{1}{3}\cdot\frac{4^{3/2}}{3/2}$$

$$= \frac{250}{9} - \frac{16}{9} = \frac{234}{9} = 26$$

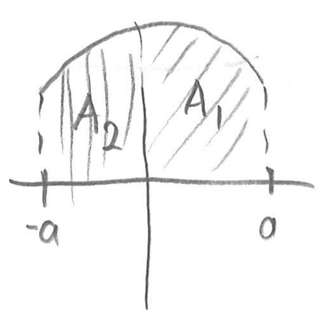
Integrals of symmetric function

Suppose f is continuous on $[-a,a]$

a) If f is even, $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

b) If f is odd, $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$

IDEA



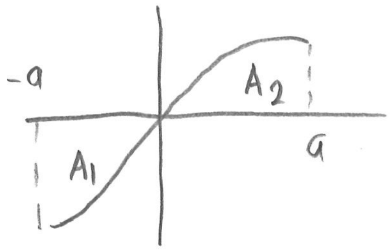
So since f is even, $A_1 = A_2$

$$\int_0^a f(x)\,dx = A_1$$

$$\int_{-a}^{a} f(x)\,dx = A_1 + A_2 = A_1 + A_2 = 2A_1$$

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

If f is odd



Then

$$\int_{-a}^{a} f(x)\,dx = A_2 - A_1 = 0.$$

<u>Ex</u> $f(x)$ be an odd function continuous on $[-2, 3]$.

If $\int_{2}^{3} f(x)\,dx = 6$, Find $\int_{-2}^{3} f(x)\,dx$

<u>Soln</u> $$\int_{-2}^{3} f(x)\,dx = \int_{-2}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx$$

$$= 0 + 6 = 6.$$

<u>Ex</u> $$\int_{-1}^{1} \frac{\sin x}{1+x^2}\,dx$$

$$f(x) = \frac{\sin x}{1+x^2}$$

$$f(-x) = \frac{\sin(-x)}{1+(-x)^2} = -\frac{\sin x}{1+x^2}$$

$$= -f(x).$$

Since f is odd,

integral is 0.